JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 130, 334–343 (1988)

Equivalence of Saddle-Points and Optima, and Duality for a Class of Non-smooth Non-convex Problems*

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Submitted by E. Stanley Lee

Received May 5, 1986

The equivalence between saddle-points and optima, and duality theorems are established for a much larger class of non-smooth non-convex problems in which functions are locally Lipschitz and are satisfying invex-type conditions of Hanson and Craven. © 1988 Academic Press, Inc.

1. INTRODUCTION

For an inequality constrained minimization problem, a saddle-point of the Lagrangian is always a (global) minimum of the problem. It is well known that under convexity assumption and a regularity hypothesis (known as constraint qualification), the two are equivalent (e.g., see Mangasarian [16]). This result plays an important role in economics and optimization theory. Various classes of non-convex (non-concave) problems have been considered for the purpose of weakening this limitation of convexity in this result. Recently, Heal [11] discussed this result for differentiable convex (concave) transformable problems; whereas, in [12], the author established the result for non-differentiable convexlike problems.

On the other hand, another basic result in nonlinear programming theory is the customary duality theorem, which asserts that, given a (primal) convex minimization problem satisfying a constraint qualification, the infimal value of the primal problem cannot be smaller than the supremal value of the associated (dual) maximization problem, and the optimal values of the primal and the dual problems are equal. This result is often useful in some computational applications where the choice is often made so that evaluating the dual maximum is significantly easier than solv-

* This work was partially produced while the author was a Ph. D. student at the University of Melbourne under the supervision of Dr. B. D. Craven, whose helpful guidance is much appreciated.

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0022-247X/88 $3.00

ing a primal minimization problem. Over the years, many generalizations of this result to non-differentiable convex problems (e.g., Schechter [21]) and differentiable non-convex problems (e.g., Hanson [9], Mond and Weir [18]) have been given in the literature.

In this paper, it is shown that the equivalence between saddle-points and optima is not limited to convex or convex transformable differentiable problems, but continues to hold for a much wide class of non-smooth non-convex problems in which functions are locally Lipschitz and are satisfying some invex type conditions of Hanson [9] and Craven [6]. Moreover, it is shown that duality theorems of Wolfe type [23] hold for this class of problems. These results include a different duality theorem for which a dual problem is formulated using generalized Fritz John conditions of Clarke [3], rather than generalized Kuhn–Tucker conditions. This theorem avoids the usual assumption of a constraint qualification.

2. Definition and Classes of Invex Functions

Hanson [9], recently introduced into optimization theory a broad generalization of convexity for differentiable functions on R^n, that for some vector function η: $R^n \times R^n \to R^n$, the real function f satisfies, for each x, $u \in R^n$, $f(x) - f(u) \geqslant \nabla f(u)^T \eta(x, u)$, and showed that both weak duality and Kuhn–Tucker sufficiency results, in constrained optimization, hold with the generalized convexity conditions, called *invex* by Craven [6]. Then, further properties and applications of invexity for some more general problems were studied by Ben-Israel and Mond [1], Craven and Glover [7], Hanson and Mond [10], Martin [17], and others.

Following Hanson [9], and Jeyakumar [13], in this section, the notion of invexity is now further generalized, for locally Lipschitz functions, to the notion called *ρ-invex* (see Definition 2.1), in which the defining inequality for invex holds approximately, to within a term depending on a parameter ρ which may be zero (invex), positive (strongly invex), or negative (weakly invex), and various classes of such functions are examined.

We begin by fixing some preliminary results that will be used. Let Ω be an open subset of R^n. A real valued function h: $\Omega \to R$ is said to be *locally lipschitz* if there exists a positive constant k such that

$$(\forall x,\ y \in \Omega)\ |h(x) - h(y)| \leqslant k\ \|x - y\|.$$

For a local Lipschitz function h, the *Clarke generalized directional derivative* and the *generalized subdifferential* are, respectively, defined by

$$h^0(a, x) := \limsup_{d \to 0, \lambda \downarrow 0} \lambda^{-1}[h(a + d + \lambda x) - h(a + d)]$$

$$\partial^0 h(a) := \{v \in R^n\colon h^0(a, x) \geqslant \langle v, x\rangle, \forall x \in \mathbb{R}^n\}.$$

Clarke [4] has shown that, when the function h is convex, $\partial^0 h(a)$ reduces to the subdifferential, $\partial h(a)$, in the sense of convex analysis (see Rockafellar [19]), when it is continuously differentiable, $\partial^0 h(a)$ reduces to $\{\nabla h(a)\}$.

DEFINITION 2.1. A locally Lipschitz function $h\colon \Omega \to R$ is called *ρ-invex at* $u \in \Omega$, with respect to some functions η, $\theta\colon \Omega \times \Omega \to R^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, if there exists a real number ρ such that for each $\xi \in \partial^0 h(u)$,

$$h(x) - h(u) \geqslant \langle \xi, \eta(x, u) \rangle + \rho \, \|\theta(x, u)\|^2, \qquad \forall x \in \Omega \tag{2.1}$$

The function is called ρ-invex, if (2.1) holds for each $u \in \Omega$. If $\rho > 0$, then h is said to be *strongly invex*. If $\rho = 0$, then h is said to be *invex* (cf. [9, 6]). If $\rho < 0$, then h is said to be *weakly invex*. It is clear that strongly invex $\Rightarrow$ invex $\Rightarrow$ weakly invex.

The classes of ρ-invex functions are given by the following propositions.

PROPOSITION 2.1. *Let $h\colon R^n \to R$ be a locally Lipschitz ρ-convex [Vial, 22] function. Then, h is ρ-invex with respect to the functions η and θ defined by $\eta(x, u) = x - u = \theta(x, u)$.*

Proof. Let x, $u \in R^n$. From Proposition 4.8 of Vial [22], for each $\xi \in \partial^0 h(u)$,

$$h(x) - h(u) \geqslant \langle \xi, (x - u) \rangle + \rho \, \|x - u\|^2,$$

and hence, h is ρ-invex.

PROPOSITION 2.2. *Let $h\colon \Omega \to R$ be a differentiable pseudo-convex (Mangasarian, [16]) function. Then, h is ρ-invex for each $\rho \geqslant 0$.*

Proof. Let x, $u \in \Omega$. Since h is pseudo-convex,

$$\nabla h(u)^T (x - u) \geqslant 0 \Rightarrow h(x) \geqslant h(u).$$

Define $\eta\colon \Omega \times \Omega \to R^n$ by

$$\eta(x, u) = \begin{cases} (x - u) & \text{if } \nabla h(u)^T (x - u) = 0 \\ (h(x) - h(u))(x - u)/\nabla h(u)^T (x - u) & \text{if } \nabla h(u)^T (x - u) \neq 0. \end{cases}$$

Hence, h is invex with respect to the above function η, and so is ρ-invex for each $\rho \geqslant 0$.

PROPOSITION 2.3. *Let the function $g\colon R^n \to R$ be locally Lipschitz and ρ-convex; let $\varphi\colon R^n \to R^n$ be continuously differentiable bijective function. If,* *for each* $u \in R^n$, $\nabla\varphi(u)$ *is onto, then the composite function* $h := g \circ \varphi$ *is* ρ*-invex with the function* θ: $R^n \times R^n \to R^n$, *defining by* $\theta(x, u) = (\varphi(x) - \varphi(u))$; $x, u \in R^n$.

Proof. Let x, $u \in R^n$ and $\xi \in \partial^0 h(u)$; let $y = \varphi(u)$ and let $z = \varphi(x)$. From the generalized chain rule [Clarke 4, Theorem 2.3.10] for differentiation, $\partial^0 h(u) = \partial^0 g(\varphi(u)) \circ \nabla\varphi(u)$; thus, ξ can be represented as a composition of γ in $\partial^0 g(\varphi(u)$ and $\nabla\varphi(u)$. Since g is ρ-convex,

$$\begin{aligned} h(x) - h(u) &= g \circ \varphi(x) - g \circ \varphi(u) \\ &= g(\varphi(x)) - g(\varphi(u)) \\ &= g(z) - (g(y) \\ &\geqslant \langle \gamma, z - y \rangle + \rho \, \|z - y\|^2 \quad \text{by Proposition 4.8 of Vial [22]).} \end{aligned}$$

Since $\nabla\varphi(u)$ is onto, $\nabla\varphi(u)^T \eta(x, u) = (z - y)$ is solvable for η. Therefore,

$$\begin{aligned} h(x) - h(u) &\geqslant \langle \gamma, \nabla\varphi(u)^T \eta(x, u) \rangle + \rho \, \|\varphi(x) - \varphi(u)\|^2 \\ &= \langle \xi, \eta(x, u) \rangle + \rho \, \|\varphi(x) - \varphi(u)\|^2. \quad \blacksquare \end{aligned}$$

EXAMPLE 2.1. A typical example of a differentiable ρ-invex function that is not ρ-convex is the function h: $R^2 \to R$, defined by $h := g \circ \varphi$, where $g(x, y) = 3x^2 - 2xy + 2y^2 + \rho(x^2 + y^2)$, and $\varphi(x, y) = (x + x^3, y + y^3)$.

Remark 2.1. Proposition 2.3 shows that *convex transformable functions in the sense of Heal* [11, p. 402] are invex functions (where $\rho = 0$). It should be noted that if, for $i = 1, 2$, g_i: $R^n \to R$ are locally Lipschitz and ρ_i-convex functions, and if φ: $R^n \to R^n$ is continuously Fréchet differentiable, bijective, and for each $u \in R^n$, $\nabla\varphi(u)$ is onto, then the composite functions $h_1 := g_1 \circ \varphi$, and $h_2 := g_2 \circ \varphi$ are ρ_1-invex and ρ_2-invex respectively, with respect to some functions η and θ which are the *same* for h_1 and h_2.

3. SADDLE-POINT THEOREMS

A fundamental result of optimization theory is that a saddle-point of the Lagrangian is equivalent to an optimum of the associated convex programming problem satisfying a constraint qualification. The significance of this result in economics has been widely demonstrated in the literature (e.g., see Heal [11, Sect. 6], and other references therein). Over the years, various generalizations of this result have been established to non-convex programming problems (e.g., see Ben-Israel and Mond [1], Jeyakumar [12], and Rockafellar [20]). More recently, Heal [11] extended the result to (concave) convex-transformable differentiable problems which are not necessarily (concave) convex, but are equivalent to (concave) convex problems up to a diffeomorphism.

In this section, it is shown that the equivalence between saddle-points and optima holds for a much large class of non-differentiable non-convex problems in which functions are locally Lipschitz and are satisfying invex type conditions. The result is proved using the Clarke necessary optimality conditions [3].

Consider the problem

(P) Minimize $f(x)$ subject to

$$x \in R^n,\ g_i(x) \leqslant 0,\ i = 1, 2, ..., m,$$

where f: $R^n \to R$ and g_i: $R^n \to R$, $i = 1, 2, ..., m$, are locally Lipschitz functions.

For the problem (P), the point (x, λ) is said to be a *critical point* if, x is a feasible point for (P), $\lambda \in \mathbb{R}^m_+$,

$$0 \in \partial^0 f(x) + \sum \lambda_i\, \partial^0 g_i(x) \qquad \text{and} \qquad \lambda_i\, g_i(x) = 0. \qquad \text{(GKT)}$$

Suppose that the problem (P) attains a local minimum at $x = a \in R^n$, then the following generalized Fritz John conditions hold:

$$(\exists \tau \in R_+,\ \lambda \in R^m_+,\ (\tau, \lambda) \neq (0, 0))$$

$$0 \in \tau\, \partial^0 f(a) + \sum \lambda_i\, \partial^0 g_i(a) \quad \text{and} \quad \lambda_i\, g_i(a) = 0, \text{ for } i = 1, 2, ..., m. \qquad \text{(GFJ)}$$

If, in addition, a suitable constraint qualification holds [see Clarke 4, pp. 169–172] for (P) then there exists $\lambda^- \in R^m_+$ such that (a, λ^-) is a critical point for (P).

THEOREM 3.1 (Saddle-point conditions). *For the locally Lipschitz problem* (P), *let the function f be ρ_0-invex and let g_i, $i = 1, 2, ..., m$, be ρ_i-invex with respect to the same functions η and θ. Suppose that (a, λ^-) is a critical point for* (P), *and that $(\rho_0 + \sum \lambda_i^- \rho_i) \geqslant 0$. Then, a is a global minimum for* (P), *and (a, λ^-) is a saddle-point of the Lagrangian*; *thus,*

$$(\forall x \in R^n)(\forall \lambda \in R^m_+) \qquad L(a, \lambda) \leqslant L(a, \lambda^-) \leqslant L(x, \lambda^-).$$

Proof. From the Generalized Kuhn–Tucker (GKT) conditions, there exist $\xi_0^- \in \partial^0 f(a)$ and $\xi_i^- \in \partial^0 g_i(a)$, for $i = 1, 2, ..., m$, such that $\xi_0^- + \sum \lambda_i^- \xi_i^- = 0$. Now, by the ρ-invexity hypotheses, for each $x \in R^n$,

$$f(x) \geqslant f(a) + \langle \xi_0^-, \eta(x, a) \rangle + \rho_0\, \|\theta(x, a)\|^2,$$

and

$$g_i(x) \geqslant g_i(a) + \langle \xi_i^-, \eta(x, a) \rangle + \rho_i \|\theta(x, a)\|^2 \qquad \text{for each } i = 1, 2, ..., m.$$

Since $\lambda_i^- \geqslant 0$,

$$f(x) + \sum \lambda_i^- g_i(x) \geqslant f(a) + \sum \lambda_i^- g_i(a) + \left\langle \left(\xi_0^- + \sum \lambda_i^- \xi_i^- \right), \eta(x, a) \right\rangle$$

$$+ \left(\rho_0 + \sum \lambda_i^- \rho_i \right) \|\theta(x, a)\|^2$$

$$= f(a) + \sum \lambda_i^- g_i(a) + \left(\rho_0 + \sum \lambda_i^- \rho_i \right) \|\theta(x, a)\|^2$$

$$\geqslant f(a) + \sum \lambda_i^- g_i(a) \qquad \text{(by the assumption)}.$$

The saddle-point condition follows from this by noting that $\sum \lambda_i g_i(a) \leqslant 0$, for each $\lambda \geqslant 0$. The global optimality follows by observing that $\sum \lambda_i^- g_i(x) \leqslant 0$, for each feasible point of (P), and $\sum \lambda_i^- g_i(a) = 0$. ∎

THEOREM 3.2 (Equivalence of saddle-point and minima). *For the locally Lipschitz problem* (P), *let the function f be ρ_0-invex and let g_i, $i = 1, 2, ..., m$, be ρ_i-invex with respect to the same functions η and θ. Suppose that the calmness constraint qualification of Clarke is satisfied and that, for each critical point (x, λ) of* (P), *$(\rho_0 + \sum \lambda_i \rho_i) \geqslant 0$. Then, the point a is a global minimum for* (P) *if and only if there exists $\lambda^- \in R_+^m$ such that (a, λ^-) forms the saddle-point conditions,*

$$(\forall x \in R^n)(\forall \lambda \in R^m, \lambda \geqslant 0), \qquad L(a, \lambda) \leqslant L(a, \lambda^-) \leqslant (x, \lambda^-). \tag{3.1}$$

Proof. Assume that a is a global minimum for (P). Then, from a theorem of Clarke [3], there exists $\lambda^- \in R_+^m$, such that (a, λ^-) is a critical point for (P). Therefore, by theorem 3.1, the saddle-point conditions (3.1) hold.

To prove the sufficiency, assume that the saddle-point conditions hold. From the left inequality of (3.1), for all $\lambda \in R_+^m$, $\sum \lambda_i g_i(a) \leqslant \sum \lambda_i^- g_i(a)$. So, in particular, for all $\lambda' \in R_+^m$, $\sum (\lambda_i' + \lambda_i^-) g_i(a) \leqslant \sum \lambda_i^- g_i(a)$; thus, for all $\lambda' \in R_+^m$, $\sum \lambda_i' g_i(a) \leqslant 0$. Hence, $g_i(a) \leqslant 0$, $i = 1, ,2, ..., m$. Since $0 \in R_+^m$, $\sum \lambda_i^- g_i(a) \geqslant 0$, so, $\sum \lambda_i^- g_i(a) = 0$. Therefore, the right inequality of the saddle-point condition (3.1) implies that a is a global minimum of (P). ∎

Remark 3.2. The assumptions in Theorem 3.1 do not demand that the functions f and g_i, $i = 1, 2, ..., m$, are convex, convex transformable, or invex, rather the theorem requires that the Lagrangian is invex. In other words, if the constraints are weakly invex then the objective function f should be strongly invex to have the saddle-point conditions.

4. Duality Theorems

In this section, two different duality theorems of Wolfe type are established for the class of locally Lipschitz problems satisfying invexity conditions.

Duality results are usually obtained using conventional Kuhn–Tucker type conditions and convexity assumptions (e.g., see Craven [5], and Schechter [21]), and therefore a constraint qualification is required to prove a strong duality theorem. This situation may be improved by either of two approaches. One is to use suitably modified Kuhn–Tucker conditions, obtained by Borwein and Wolkowicz [2] without the use of a constraint qualification. This has been treated recently by Kanniappan [15].

The second approach is to use conventional Fritz John-type conditions, instead of Kuhn–Tucker-type conditions. The idea of using Fritz John-type conditions to prove duality theorems for problems (primal and dual), whose objective functions are of the same form, was first noticed by Dantzig, Eisenberg, and Cottle [8]. Very recently, in Jeyakumar, Weir, and Mond [14], this idea was used to prove strong duality theorems for differentiable programming problems.

In the next theorem, a duality result is established using Generalized Fritz John conditions under a strengthened invexity assumptions. Hence, at the expense of a strengthened invexity assumption, the usual assumption of a constraint qualification is dropped.

We associate to the problem (P), the following dual problem

$$\text{(D1)}\quad \text{Maximize}\, f(\xi) \text{ subject to } 0 \in \tau\, \partial^0 f(\xi) + \sum \lambda_i\, \partial^0 g_i(\xi),\ \lambda^T g(\xi) \geqslant 0,$$
$$\tau \geqslant 0,\ \lambda \geqslant 0,\ (\tau, \lambda) \neq (0, 0).$$

We observe that the primal and the dual problems have the same form of objective function.

Theorem 4.1 (Strong and converse duality theorem). *Consider the problems* (P) *and* (D1). *Let the functions f and g_i, $i = 1, 2, ..., m$, be locally Lipschitz. Suppose that the point a is optimal for* (P). *If f is ρ_0-invex and, for each $i = 1, 2, ..., m$, g_i is ρ_i-invex, with respect to the same functions η and θ, and if $(\tau\rho_0 + \sum \lambda_i \rho_i) > 0$, for each feasible point (ξ, τ, λ) of* (D1), *then there exist $\tau^- \geqslant 0$, and $\lambda^- \geqslant 0$, such that (a, τ^-, λ^-) is* (*global*) *optimal for* (D1) *and the optimal values of* (P) *and* (D1) *are equal. Moreover, if $(\xi_0, \tau_0, \lambda_0)$ is another* (*global*) *optimal solution for* (D1) *then $a = \xi_0$; that is, ξ_0 solves the problem* (P).

Remark 4.1. It should be noted that Theorem 4.1 requires that the Lagrangian function, $\tau f + \sum \lambda_i g_i$, is strongly invex, which would be satisfied if, in particular, the functions are strongly invex with respect to the same functions η and θ. However, a constraint qualification is not assumed here. Moreover, Theorem 4.1 includes a corresponding strict converse duality result.

Proof. To prove a weak duality result, let x be feasible for (P) and (ξ, τ, λ) be feasible for (D1). Then there exist $v \in \partial^0 f(\xi)$ and $w_i \in \partial^0 g_i(\xi)$ such that $\tau v + \sum \lambda_i w_i = 0$. If $x = \xi$, then a weak duality trivially holds, so, we assume that $x \neq \xi$. Suppose that $f(x) < f(\xi)$. Then,

$$
\begin{aligned}
0 &\geqslant \tau f(x) - \tau f(\xi) && (\text{since } \tau \geqslant 0)\\
&\geqslant \tau\langle v, \eta(x, \xi)\rangle + \tau\rho_0 \|\theta(x, \xi)\|^2 && (\text{by } \rho_0\text{-invexity})\\
&\geqslant -\left\langle \sum \lambda_i w_i, \eta(x, \xi)\right\rangle + \tau\rho_0 \|\theta(x, \xi)\|^2 &&\\
&\geqslant -\lambda^T g(x) + \lambda^T g(\xi) + \left(\tau\rho_0 + \sum \lambda_i \rho_i\right) \|\theta(x, \xi)\|^2 && (\text{by } \rho_i\text{-invexity})\\
&\geqslant \left(\tau\rho_0 + \sum \lambda_i \rho_i\right) \|\theta(x, \xi)\|^2 && (\text{by feasibility})\\
&> 0 \qquad \left(\text{since } x \neq \xi, \text{ and } \left(\tau\rho_0 + \sum \lambda_i \rho_i\right) > 0\right), &&
\end{aligned}
$$

a contradiction, and hence $f(x) \geqslant f(\xi)$.

Now, since a is optimal for (P), there exist $\tau^- \geqslant 0$, and $\lambda^- \geqslant 0$, such that $0 \in \tau^- \partial^0 f(a) + \sum \lambda_i^- \partial^0 g_i(a)$, and $\lambda^{-T} g(a) = 0$. Therefore, (a, τ^-, λ^-) is feasible for (D1). This with weak duality shows that (a, τ^-, λ^-) is optimal for (D1) and the optimal values of (P) and (D1) are equal.

To prove $a = \xi_0$, we assume that $a \neq \xi_0$ and exhibit a contradiction. Since (a, τ^-, λ^-) and $(\xi_0, \tau_0, \lambda_0)$ are optimal for (D1), there exist $v_0 \in \partial^0 f(\xi_0)$ and $w_{0i} \in \partial^0 g_i(\xi_0)$ such that $\tau_0 v_0 + \sum \lambda_{0i} w_{0i} = 0$, and $f(a) = f(\xi_0)$. Then, by the same arguments as above, $0 = \tau_0 f(a) - \tau_0 f(\xi_0) > 0$, a contradiction. ∎

In the next theorem, we obtain a duality result using generalized Kuhn–Tucker conditions, extending a result of Schechter [21], and Wolfe [23]. Here, invexity assumptions are reduced; however, a constraint qualification is implicitly assumed (by a critical point assumption).

Consider the following dual problem

$$
\text{(D2)} \quad \text{Maximize } f(\xi) + \sum \lambda_i g_i(\xi) \text{ subject to } 0 \in \partial^0 f(\xi) + \sum \lambda_i \partial^0 g_i(\xi), \quad \lambda \geqslant 0.
$$

THEOREM 4.2. *Consider the problems* (P) *and* (D2). *Let the functions f and g_i, $i = 1, 2, \ldots, m$, be locally Lipschitz. Assume that the point (a, λ^-) is a* *critical point for* (P). *If* f *is* ρ_0*-invex and, for each* $i=1, 2, ..., m$, g_i *is* ρ_i*-invex, with respect to the same functions* η *and* θ, *and if,* $(\rho_0 + \sum \lambda_i \rho_i) \geqslant 0$, *for each feasible point* (ξ, λ) *of* (D2), *then* a *is global optimal for* (P), (a, λ^-) *is global optimal for* (D2), *and the optimal values of* (P) *and* (D2) *are equal.*

Proof. We first prove the weak duality result. Let x be feasible for (P) and let (ξ, λ) be feasible for (D2). Then, there exist $v \in \partial^0 f(\xi)$ and $w_i \in \partial^0 g_i(\xi)$ such that $v + \sum \lambda_i w_i = 0$. Now, by ρ-invexity assumptions,

$$\begin{aligned}
& f(x) - [f(\xi) + \lambda^T g(\xi)] \\
& \quad \geqslant \langle v. \eta(x, \xi) \rangle + \rho_0 \|\theta(x, \xi)\|^2 - \lambda^T g(\xi) \\
& \quad = \left\langle -\sum \lambda_i w_i, \eta(x, \xi) \right\rangle + \rho_0 \|\theta(x, \xi)\|^2 - \lambda^T g(\xi) \\
& \quad \geqslant -\lambda^T g(x) + \left(\rho_0 + \sum \lambda_i \rho_i \right) \|\theta(x, \xi)\|^2 \\
& \quad \geqslant 0,
\end{aligned}$$

hence, $f(x) \geqslant f(\xi) + \lambda^T g(\xi)$. This with the assumption that (a, λ^-) is a critical point for (P), gives that a and (a, λ^-) are, respectively, global optimal solutions for (P) and (D2), and the optimal values of (P) and (D2) are equal. ∎

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